Fundamentals of Electrical Engineering
(EPCE 2101)
Credit: 4Hrs (Lec.: 2 Hr, Tut:3 Hr and Lab: 3 Hr)

Chapter-5
AC Steady State Analysis

Outline:

1. Introduction Sinusoids
2. Sinusoidal and complex forcing functions
3. Phasors
4. Phasors representation for circuit elements
5. Impedance and admittance
6. Phasor diagrams
7. AC circuit analysis techniques

Tamiru Getahun (Msc.)

ASTU

1.Introduction

- We have restricted the forcing function to dc sources for the sake of simplicity, for pedagogic reasons, and also for historic reasons.
- Historically, dc sources were the main means of providing electric power up until the late 1800s.
- At the end of that century, the battle of direct current versus alternating current began.
- But, ac is more efficient and economical to transmit over long distances, ac systems ended up the winner.
- we are particularly interested in sinusoidally time-varying excitation, or simply, excitation by a sinusoid.

- Circuits driven by sinusoidal current or voltage sources are called **ac circuits.**
- The Reason why we are interested in sinusoids is
    - I. Nature itself is characteristically sinusoidal.
    - II. sinusoidal signal is easy to generate and transmit.
    - III. any practical periodic signal can be represented by a sum of sinusoid.
    - IV. a sinusoid is easy to handle mathematically.

Sinusoids

Consider the sinusoidal voltage

$$v(t) = V_m \sin \omega t$$

where

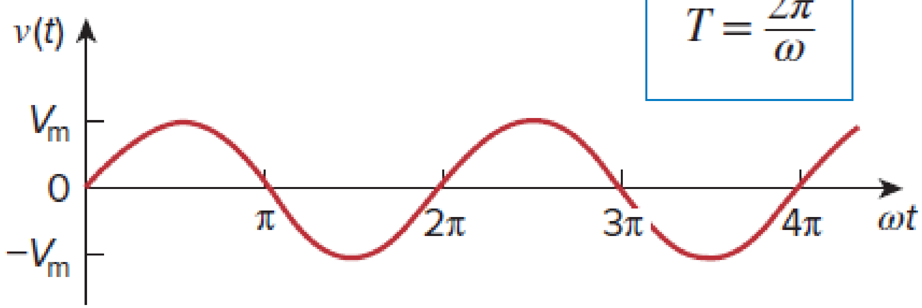
V_m = the *amplitude* of the sinusoid

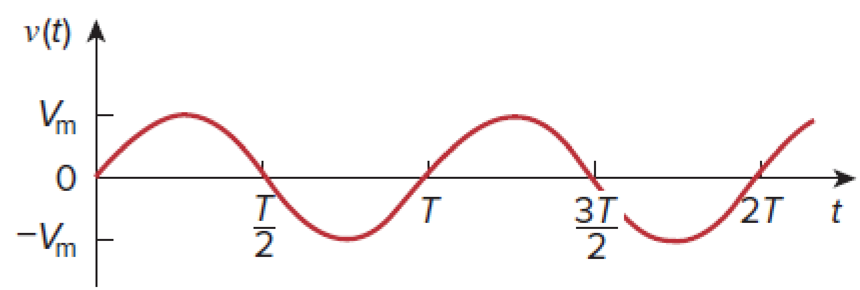
ω = the *angular frequency* in radians/s

ωt = the *argument* of the sinusoid

- ***T*** is called the *period* of the sinusoid. From the two plots in Fig. below, we observe that $\omega T = 2\pi$,

$$T = \frac{2\pi}{\omega}$$





- The fact that $v(t)$ repeats itself every T seconds is shown by replacing t by $t + T$

$$v(t+T) = V_m \sin \omega(t+T) = V_m \sin \omega\left(t + \frac{2\pi}{\omega}\right)$$

$$= V_m \sin(\omega t + 2\pi) = V_m \sin \omega t = v(t)$$

$$\boxed{v(t+T) = v(t)}$$

- A periodic function is one that satisfies f (t) = f (t + nT), for all t and for all integers n.
- The cyclic frequency f of the sinusoid is,

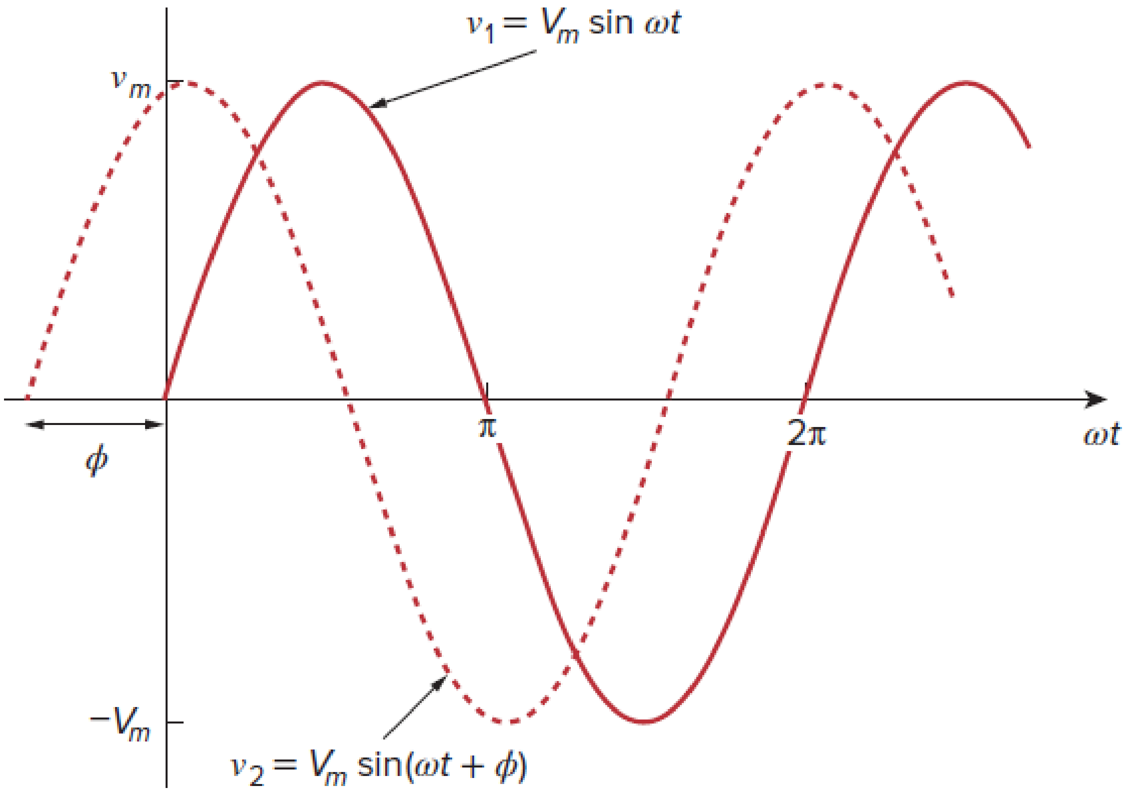
$$\boxed{f = \frac{1}{T}} \Rightarrow \omega = 2\pi f$$

- The general expression for the sinusoid $v(t) = V_m \sin(\omega t + \phi)$

Where, ϕ is the *phase*

Let us examine the two sinusoids

$$v_1(t) = V_m \sin \omega t \qquad \text{and} \qquad v_2(t) = V_m \sin(\omega t + \phi)$$



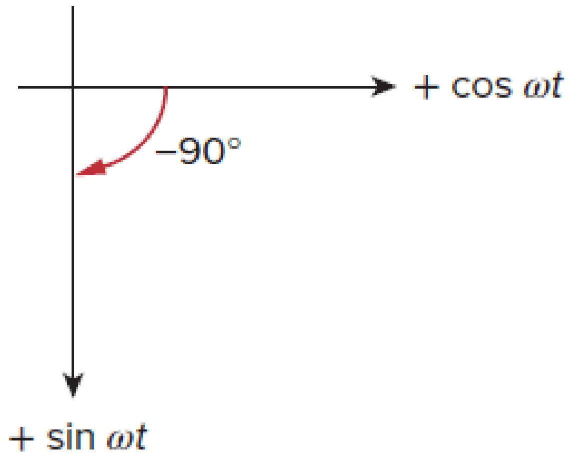
- If $\phi = 0$, then v_1 and v_2 are said to be **in phase**.
- If $\phi \neq 0$, then v_1 and v_2 are said to be **Out of phase**.
- v_2 leads v_1 by ϕ or that v_1 lags v_2 by ϕ.

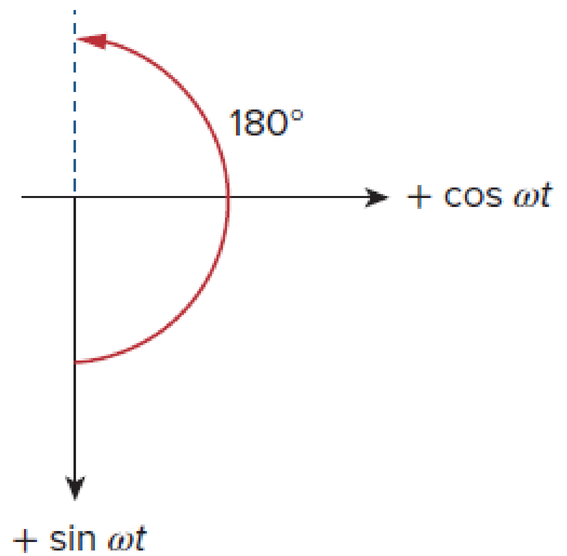
Figure: Two sinusoids with different phases

- A sinusoid can be expressed in either sine or cosine form
- When comparing two sinusoids, it is expedient to express both as either sine or cosine with positive amplitudes.
- This is achieved by using the following trigonometric identities:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(\omega t \pm 180^\circ) = -\sin \omega t$$
$$\cos(\omega t \pm 180^\circ) = -\cos \omega t$$
$$\sin(\omega t \pm 90^\circ) = \pm \cos \omega t$$
$$\cos(\omega t \pm 90^\circ) = \mp \sin \omega t$$

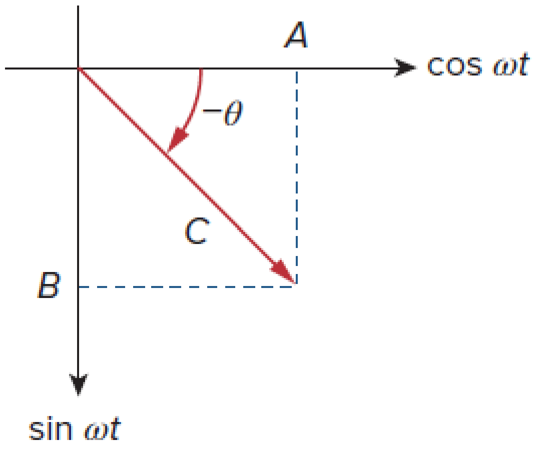




- Let as add two sinusoids of the same frequency

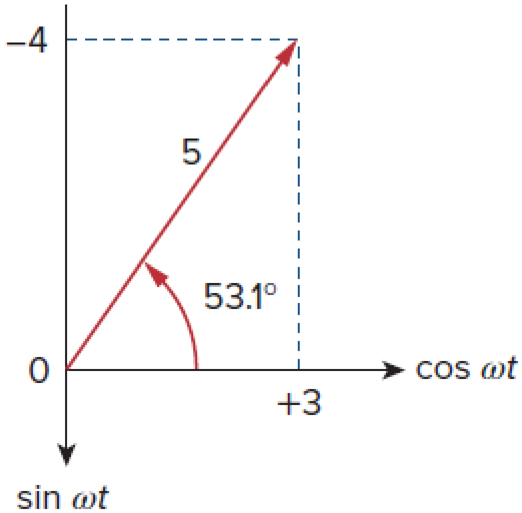
$$A\cos\omega t + B\sin\omega t = C\cos(\omega t - \theta)$$

where $C = \sqrt{A^2 + B^2}, \qquad \theta = \tan^{-1}\frac{B}{A}$



Example:

$$3\cos\omega t - 4\sin\omega t = 5\cos(\omega t + 53.1°)$$



Example: Find the amplitude, phase, period, and frequency of the sinusoid

$$v(t) = 12\cos(50t + 10^\circ)\text{ V.}$$

Solution:

The amplitude is $V_m = 12$ V.
The phase is $\phi = 10^\circ$.
The angular frequency is $\omega = 50$ rad/s.
The period $T = \frac{2\pi}{\omega} = \frac{2\pi}{50} = 0.1257$ s.
The frequency is $f = \frac{1}{T} = 7.958$ Hz.

Example: Calculate the phase angle between V_1 & V_2. Then, State which sinusoid is leading.

$$v_1 = -10\cos(\omega t + 50^\circ)$$

$$v_2 = 12\sin(\omega t - 10^\circ)$$

Solution:

METHOD 1

In order to compare *v*1 and *v*2, we must express them in the same form.

$$v_1 = -10\cos(\omega t + 50^\circ) = 10\cos(\omega t + 50^\circ - 180^\circ)$$

$$v_1 = 10\cos(\omega t - 130^\circ) \quad \text{or} \quad v_1 = 10\cos(\omega t + 230^\circ)$$

$$v_2 = 12\sin(\omega t - 10^\circ) = 12\cos(\omega t - 10^\circ - 90^\circ)$$

$$v_2 = 12\cos(\omega t - 100^\circ)$$

- From the above two eqs. the phase difference between v_1 and v_2 is 30°.

$$v_2 = 12\cos(\omega t - 130^\circ + 30^\circ) \quad \text{or} \quad v_2 = 12\cos(\omega t + 260^\circ)$$

- This shows clearly that v_2 leads v_1 by 30°.

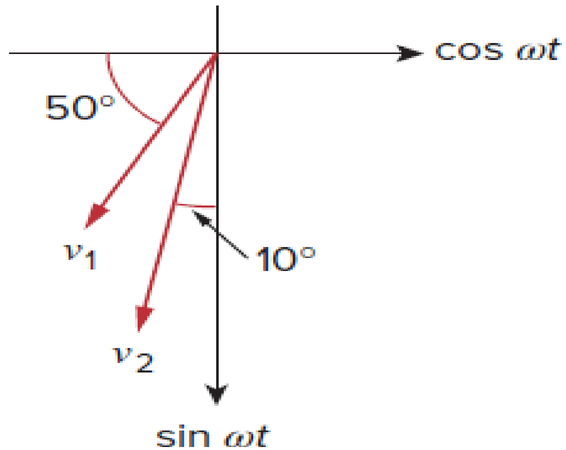
METHOD 2: Alternatively, we may express *v*1 in sine form:

$$v_1 = -10\cos(\omega t + 50^\circ) = 10\sin(\omega t + 50^\circ - 90^\circ)$$

$$= 10\sin(\omega t - 40^\circ) = 10\sin(\omega t - 10^\circ - 30^\circ)$$

But $v_2 = 12\sin(\omega t - 10°)$. Comparing the two shows that v_1 lags v_2 by 30°. This is the same as saying that v_2 leads v_1 by 30°.

- **METHOD 3:** If you plot simply plot the graph for v1 and v2



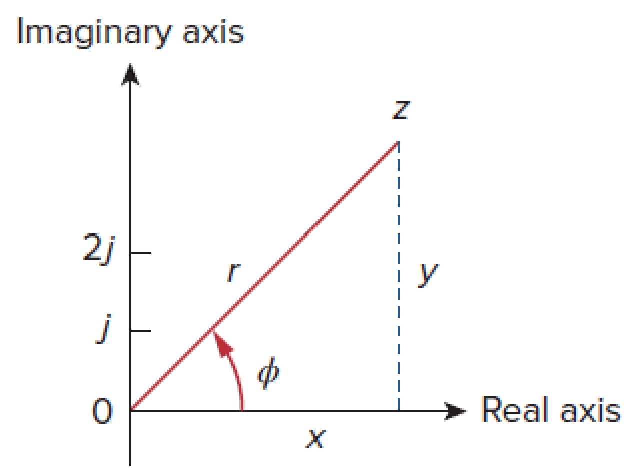
Phasors

- A phasor is a complex number that represents the amplitude and phase of a sinusoid

$z = x + jy$ Rectangular form

$z = r\underline{/\phi}$ Polar form

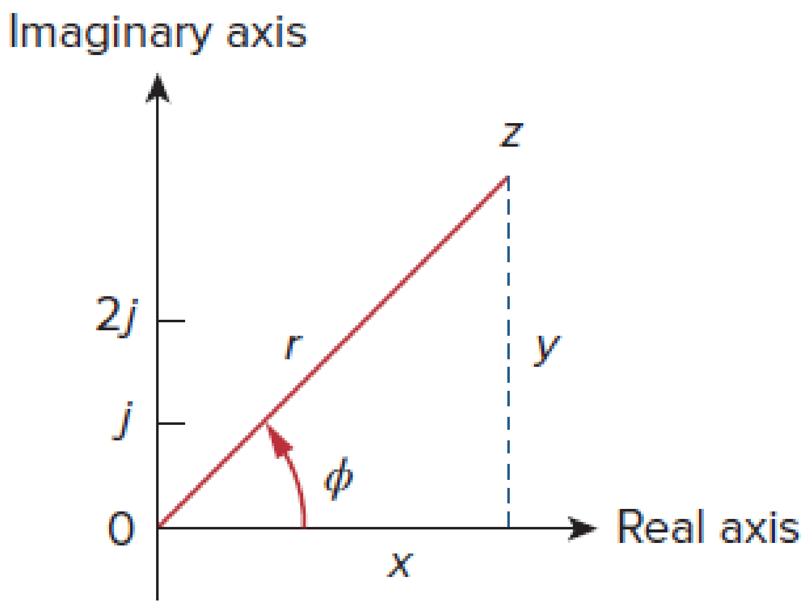
$z = re^{j\phi}$ Exponential form



$z = x + jy$ Rectangular form

$z = r\angle\phi$ Polar form

$z = re^{j\phi}$ Exponential form



Where:

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\phi, \qquad y = r\sin\phi$$

$$z = x + jy = r\angle\phi = r(\cos\phi + j\sin\phi)$$

❖ Important Operation in complex function(Basic properties)

$$z_1 = x_1 + jy_1 = r_1\angle\phi_1$$

$$z_2 = x_2 + jy_2 = r_2\angle\phi_2$$

Addition: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Subtraction: $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$

Multiplication: $z_1 z_2 = r_1 r_2 \underline{/\phi_1 + \phi_2}$

Division: $\frac{z_1}{z_2} = \frac{r_1}{r_2}\underline{/\phi_1 - \phi_2}$

Reciprocal: $\frac{1}{z} = \frac{1}{r}\underline{/-\phi}$

Complex Conjugate:

$$z^* = x - jy = r\underline{/-\phi} = re^{-j\phi}$$

Square Root:

$$\sqrt{z} = \sqrt{r}\underline{/\phi/2}$$

- **Euler's identity** forms the basis of phasor notation.
- Simply stated, the identity defines the **complex exponential** $e^{j\vartheta}$ as a point in the complex plane, which may be represented by real and imaginary components:

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

$$\cos\phi = \mathrm{Re}(e^{j\phi})$$

$$\sin\phi = \mathrm{Im}(e^{j\phi})$$

$$v(t) = V_m\cos(\omega t + \phi) = \mathrm{Re}(V_m e^{j(\omega t+\phi)})$$

$$v(t) = \mathrm{Re}(V_m e^{j\phi} e^{j\omega t})$$

$$v(t) = \mathrm{Re}(\mathbf{V} e^{j\omega t})$$

$$\Rightarrow \mathbf{V} = V_m e^{j\phi} = V_m\underline{/\phi}$$

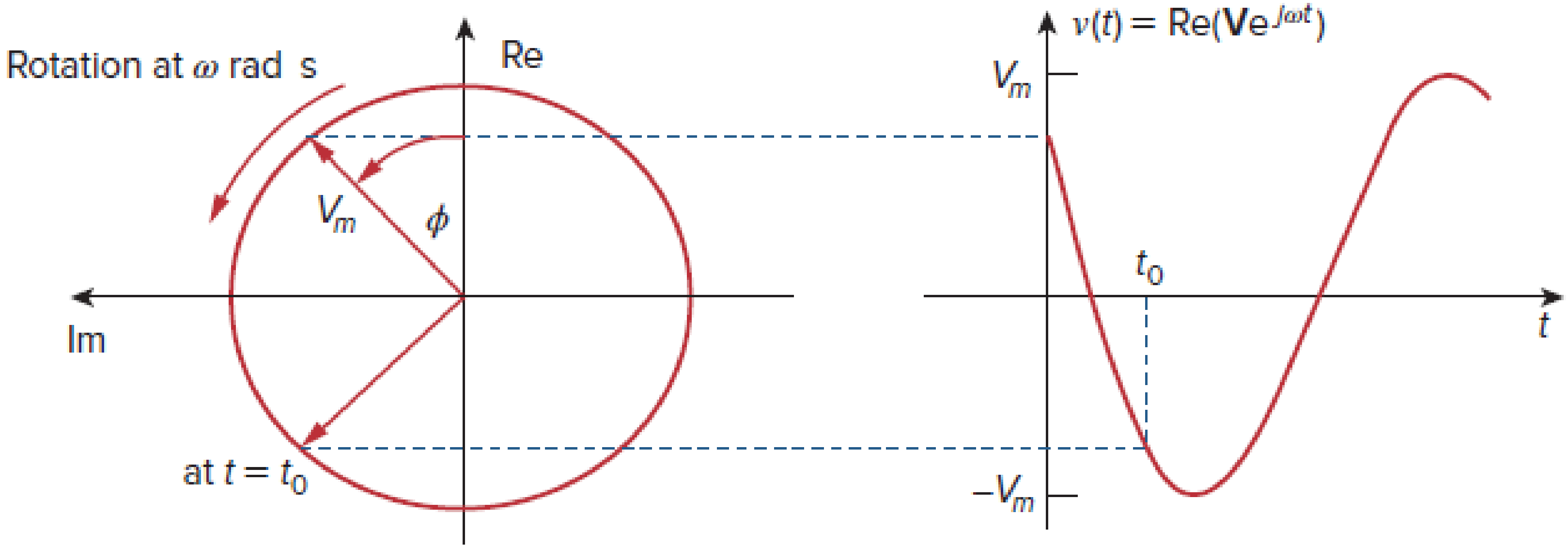


Figure: Representation of $\mathbf{V}e^{j\omega}$: (a) sinor rotating counterclockwise, (b) its projection on the real axis, as a function of time.

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)

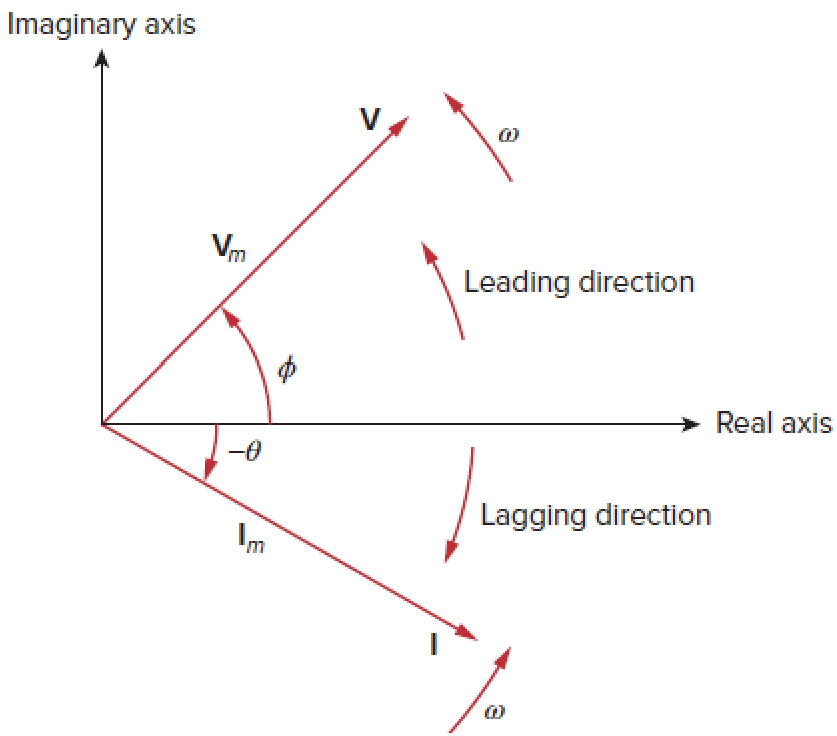


Figure: A phasor diagram showing $\mathbf{V} = V_m\underline{/\phi}$ and $\mathbf{I} = I_m\underline{/-\theta}$.

Sinusoid-phasor transformation.

Time domain representation	Phasor domain representation
$V_m\cos(\omega t + \phi)$	$V_m\underline{/\phi}$
$V_m\sin(\omega t + \phi)$	$V_m\underline{/\phi - 90^\circ}$
$I_m\cos(\omega t + \theta)$	$I_m\underline{/\theta}$
$I_m\sin(\omega t + \theta)$	$I_m\underline{/\theta - 90^\circ}$

Let differentiate V(t) w.r.t $v(t) = \text{Re}(\mathbf{V}e^{j\omega t}) = V_m \cos(\omega t + \phi),$

$$\frac{dv}{dt} = -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi + 90^\circ)$$
$$= \text{Re}(\omega V_m e^{j\omega t} e^{j\phi} e^{j90^\circ}) = \text{Re}(j\omega \mathbf{V} e^{j\omega t})$$

- This shows that the derivative *v*(*t*) is transformed to the phasor domain as $j\omega\mathbf{V}$

$$\frac{dv}{dt} \quad \Leftrightarrow \quad j\omega\mathbf{V}$$

(Time domain) (Phasor domain)

- Similarly, the integral of *v*(*t*) is transformed to the phasor domain as $\mathbf{V}/j\omega$

$$\int v\,dt \quad \Leftrightarrow \quad \frac{\mathbf{V}}{j\omega}$$

(Time domain) (Phasor domain)

The differences between *v*(*t*) and V should be emphasized:

I. v(t) is the instantaneous or time domain representation, while **V** is the frequency or phasor domain representation.

II. v(t) is time dependent, while **V** is not.

III. v(t) is always real with no complex term, while **V** is generally complex.

- **Finally,** we should bear in mind that phasor analysis applies only when frequency is constant; it applies in manipulating two or more sinusoidal signals only if they are of the same frequency.

Example: Evaluate these complex numbers

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $\dfrac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$

Solution:

(a) Using polar to rectangular transformation,

$$40\angle 50^\circ = 40(\cos 50^\circ + j\sin 50^\circ) = 25.71 + j30.64$$

$$20\angle -30^\circ = 20[\cos(-30^\circ) + j\sin(-30^\circ)] = 17.32 - j10$$

Adding them up gives

$$40\angle 50^\circ + 20\angle -30^\circ = 43.03 + j20.64 = 47.72\angle 25.63^\circ$$

Taking the square root of this,

$$(40\angle 50^\circ + 20\angle -30^\circ)^{1/2} = 6.91\angle 12.81^\circ$$

(b) Using polar-rectangular transformation, addition, multiplication, and division,

$$\frac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*} = \frac{8.66 - j5 + (3 - j4)}{(2 + j4)(3 + j5)}$$

$$= \frac{11.66 - j9}{-14 + j22} = \frac{14.73\angle -37.66^\circ}{26.08\angle 122.47^\circ}$$

$$= 0.565\angle -160.13^\circ$$

Example: Transform these sinusoids to phasors

(a) $i = 6\cos(50t - 40^\circ)$ A
(b) $v = -4\sin(30t + 50^\circ)$ V

Solution:

(a) $i = 6\cos(50t - 40^\circ)$ has the phasor

$$\mathbf{I} = 6\angle -40^\circ \text{ A}$$

(b) Since $-\sin A = \cos(A + 90^\circ)$,

$$v = -4\sin(30t + 50^\circ) = 4\cos(30t + 50^\circ + 90^\circ)$$
$$= 4\cos(30t + 140^\circ) \text{ V}$$

The phasor form of v is

$$\mathbf{V} = 4\angle 140^\circ \text{ V}$$

Example: Find the sinusoids represented by these phasors

(a) $\mathbf{I} = -3 + j4$ A

(b) $\mathbf{V} = j8e^{-j20^\circ}$ V

Solution:

(a) $\mathbf{I} = -3 + j4 = 5\underline{/126.87^\circ}$. Transforming this to the time domain gives

$$i(t) = 5\cos(\omega t + 126.87^\circ)\ \text{A}$$

(b) Because $j = 1\underline{/90^\circ}$,

$$\begin{aligned}\mathbf{V} = j8\underline{/-20^\circ} &= (1\underline{/90^\circ})(8\underline{/-20^\circ})\\ &= 8\underline{/90^\circ - 20^\circ} = 8\underline{/70^\circ}\ \text{V}\end{aligned}$$

Converting this to the time domain gives

$$v(t) = 8\cos(\omega t + 70^\circ)\text{V}$$

Example: Find their sum. $i_1(t) = 4\cos(\omega t + 30°)$ A and $i_2(t) = 5\sin(\omega t - 20°)$ A

Solution:

Here is an important use of phasors—for summing sinusoids of the same frequency. Current $i_1(t)$ is in the standard form. Its phasor is

$$\mathbf{I}_1 = 4\underline{/30°}$$

We need to express $i_2(t)$ in cosine form. The rule for converting sine to cosine is to subtract 90°. Hence,

$$i_2 = 5\cos(\omega t - 20° - 90°) = 5\cos(\omega t - 110°)$$

$$\mathbf{I}_2 = 5\underline{/-110°}$$

If we let $i = i_1 + i_2$, then

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 4\underline{/30°} + 5\underline{/-110°}$$

$$= 3.464 + j2 - 1.71 - j4.698 = 1.754 - j2.698$$

$$= 3.218\underline{/-56.97°}\text{ A}$$

Transforming this to the time domain, we get

$$i(t) = 3.218\cos(\omega t - 56.97°)\text{ A}$$

Example: Using the phasor approach, determine the current i(t) in a circuit described by the integrodifferential equation.

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75^\circ)$$

Solution:

$$4\mathbf{I} + \frac{8\mathbf{I}}{j\omega} - 3j\omega\mathbf{I} = 50\underline{/75^\circ}$$

But $\omega = 2$, so

$$\mathbf{I}(4 - j4 - j6) = 50\underline{/75^\circ}$$

$$\mathbf{I} = \frac{50\underline{/75^\circ}}{4 - j10} = \frac{50\underline{/75^\circ}}{10.77\underline{/-68.2^\circ}} = 4.642\underline{/143.2^\circ}\text{ A}$$

Converting this to the time domain,

$$i(t) = 4.642\cos(2t + 143.2^\circ)\text{ A}$$

Keep in mind that this is only the steady-state solution, and it does not require knowing the initial values.

Phasor Relationships for Circuit Elements

- If the current through a resistor *R* is

$$i = I_m \cos(\omega t + \phi),$$

$$v = iR = RI_m \cos(\omega t + \phi)$$

- The phasor form of this voltage is

$$\mathbf{V} = RI_m \angle \phi \qquad \mathbf{I} = I_m \angle \phi.$$

$$\mathbf{V} = R\mathbf{I}$$

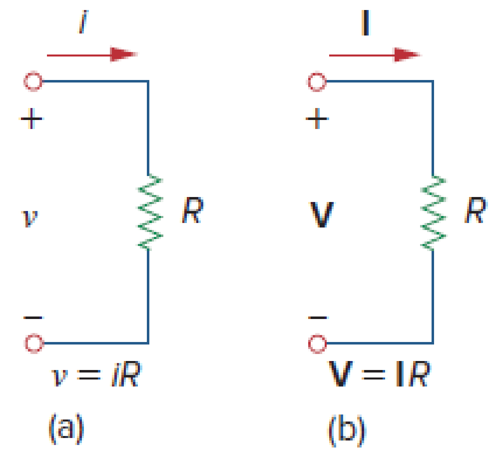


Figure: Voltage-current relations for a resistor in the: (a) time domain, (b) frequency domain.

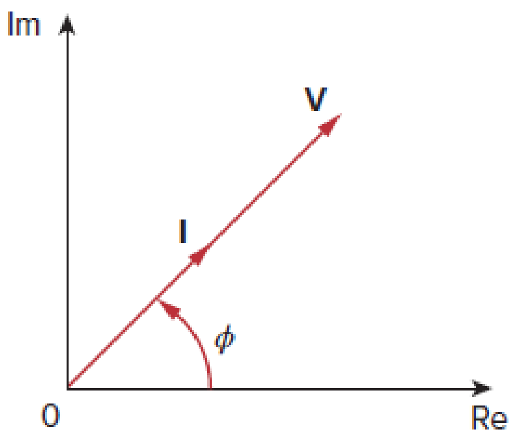


Figure: Phasor diagram for the resistor.

- For the inductor L, assume the current through it is

$$i = I_m \cos(\omega t + \phi).$$

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$

Since, $-\sin A = \cos(A + 90°)$.

$$v = \omega L I_m \cos(\omega t + \phi + 90°)$$

which transforms to the phasor

$$\mathbf{V} = \omega L I_m e^{j(\phi+90°)} = \omega L I_m e^{j\phi} e^{j90°} = \omega L I_m \underline{/\phi + 90°}$$

But $I_m \underline{/\phi} = \mathbf{I}$, $\qquad e^{j90°} = j$. Thus,

$$\mathbf{V} = j\omega L \mathbf{I}$$

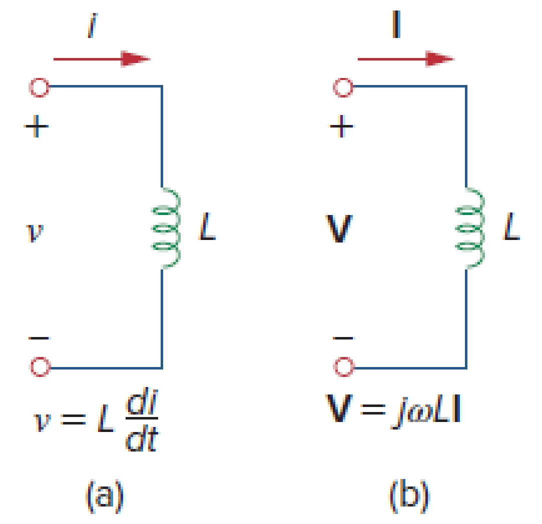


Figure: Voltage-current relations for an inductor in the: (a) time domain, (b) frequency domain

- The voltage and current are 90° out of phase.

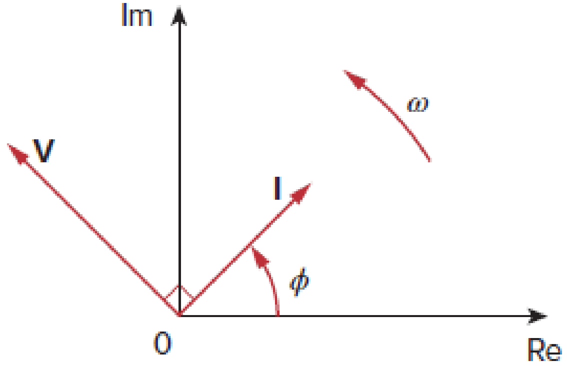


Figure: Phasor diagram for the inductor; **I** lags **V**.

- For the capacitor C, assume the voltage across it is $v = V_m \cos(\omega t + \phi)$. The current through the capacitor is

$$i = C\frac{dv}{dt}$$

$$\mathbf{I} = j\omega C\mathbf{V} \quad \Rightarrow \quad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

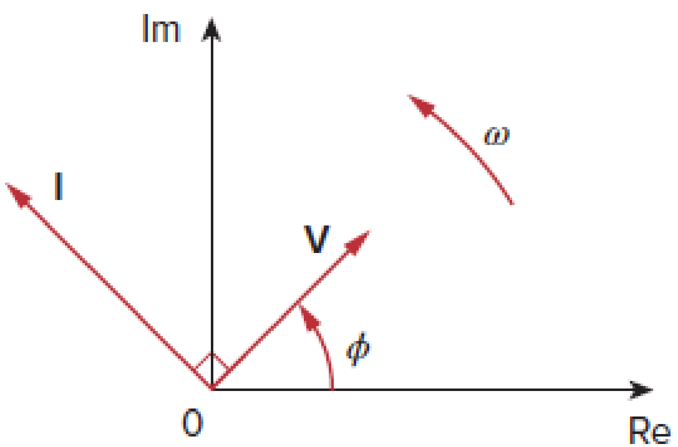


Figure: Phasor diagram for the capacitor; **I** leads **V**.

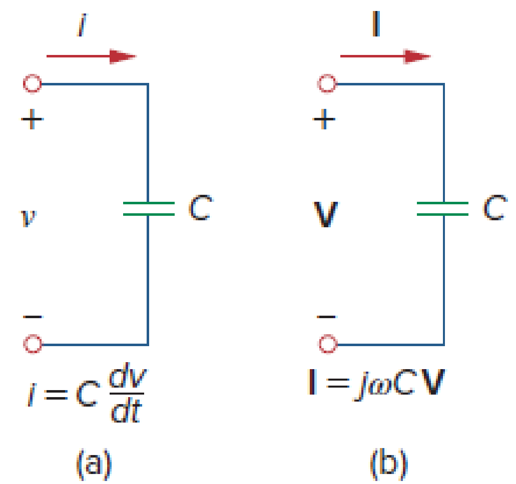


Figure: Voltage-current relations for a capacitor in the: (a) time domain, (b) frequency domain.

- The current and voltage are 90° out of phase

Summary of voltage-current relationships.

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\frac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\frac{dv}{dt}$	$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$

Example: The voltage $v = 12\cos(60t + 45°)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

Solution:

For the inductor, $\mathbf{V} = j\omega L\mathbf{I}$, where $\omega = 60$ rad/s and $\mathbf{V} = 12\,\underline{/45°}$ V. Hence,

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\,\underline{/45°}}{j60 \times 0.1} = \frac{12\,\underline{/45°}}{6\,\underline{/90°}} = 2\,\underline{/-45°}\text{ A}$$

Converting this to the time domain,

$$i(t) = 2\cos(60t - 45°)\text{ A}$$

Impedance and Admittance

In the preceding section, we obtained the voltage-current relations for the three passive elements as

$$\mathbf{V} = R\mathbf{I}, \quad \mathbf{V} = j\omega L\mathbf{I}, \quad \mathbf{V} = \frac{\mathbf{I}}{j\omega C} \Rightarrow \frac{\mathbf{V}}{\mathbf{I}} = R, \quad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \quad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

- From these three expressions, we obtain Ohm's law in phasor form for any type of element as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{or} \quad \mathbf{V} = \mathbf{Z}\mathbf{I}$$

- where **Z** is a frequency-dependent quantity known as ***impedance***, measured in ohms.

The **impedance Z** of a circuit is the ratio of the phasor voltage **V** to the phasor current **I**, measured in ohms (Ω).

- The impedance represents the opposition that the circuit exhibits to the flow of sinusoidal current.

Impedances and admittances of passive elements.

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

- As a complex quantity, the impedance may be expressed in rectangular form as

$$\mathbf{Z} = R \pm jX \qquad \mathbf{Z} = |\mathbf{Z}| \underline{/\theta}$$

where

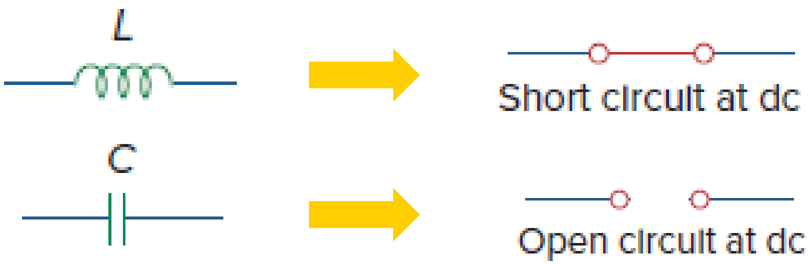
$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1}\frac{\pm X}{R}$$

and

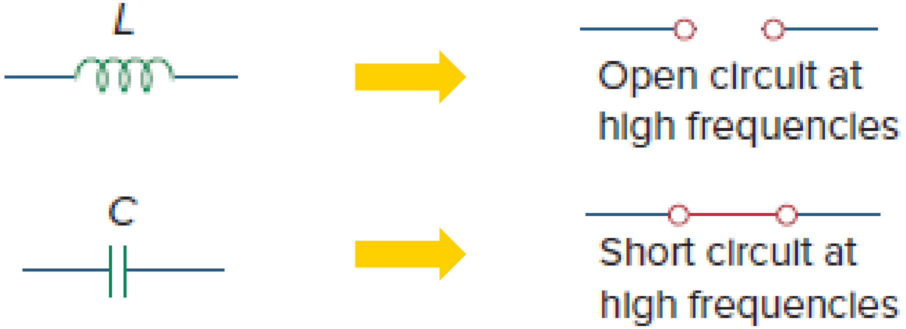
$$R = |\mathbf{Z}|\cos\theta, \qquad X = |\mathbf{Z}|\sin\theta$$

- Consider two extreme cases of angular frequency.

1. When $\omega = 0$ (i.e., for dc sources), $\mathbf{Z}_L = 0$ and $\mathbf{Z}_C \rightarrow \infty$,



2. When $\omega \rightarrow \infty$ (i.e., for high frequencies), $Z_L \rightarrow \infty$ and $Z_C = 0$,



The admittance **Y** is the reciprocal of impedance, measured in siemens (S).

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}} \quad \Rightarrow \quad \mathbf{Y} = G + jB$$

where G = Re **Y** is called the ***conductance*** and B = Im **Y** is called the ***susceptance***.

$$G + jB = \frac{1}{R + jX}$$

By rationalization

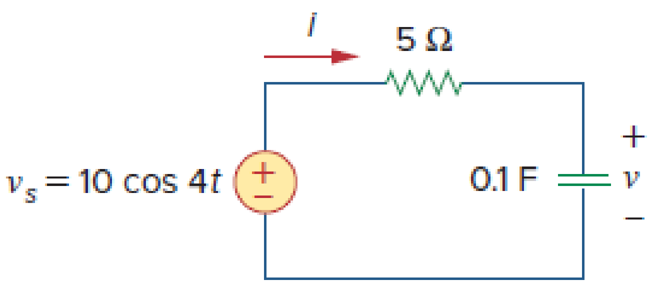
$$G + jB = \frac{1}{R + jX} \cdot \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2}$$

Equating the real and imaginary parts gives

$$G = \frac{R}{R^2 + X^2}, \qquad B = -\frac{X}{R^2 + X^2}$$

showing that $G \neq 1/R$ as it is in resisti ve circuits. Of course, if $X = 0$, then $G = 1/R$.

Example: Find v(t) and i(t) in the circuit shown in Figure below



Solution:

From the voltage source $10 \cos 4t$, $\omega = 4$,

$$\mathbf{V}_s = 10\underline{/0^\circ} \text{ V}$$

The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

Hence the current

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\underline{/0^\circ}}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2}$$

$$= 1.6 + j0.8 = 1.789\underline{/26.57^\circ} \text{ A}$$

The voltage across the capacitor is

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\underline{/26.57^\circ}}{j4 \times 0.1}$$

$$= \frac{1.789\underline{/26.57^\circ}}{0.4\underline{/90^\circ}} = 4.47\underline{/-63.43^\circ} \text{ V}$$

- Converting **I** and **V** to the time domain, we get

$$i(t) = 1.789 \cos(4t + 26.57^\circ) \text{ A}$$

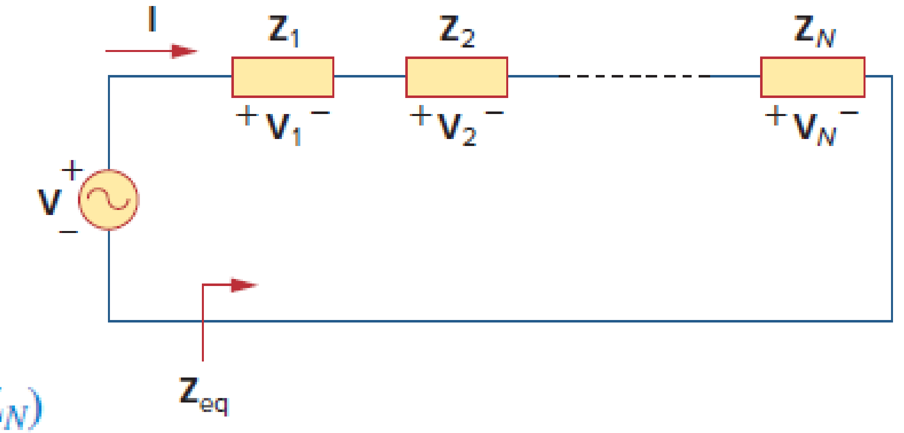
$$v(t) = 4.47 \cos(4t - 63.43^\circ) \text{ V}$$

- Notice that $i(t)$ leads $v(t)$ by 90° as expected.

Impedance Combinations

I. Impedance in Series

- The same current **I** flows through the impedances. Applying KVL around the loop gives



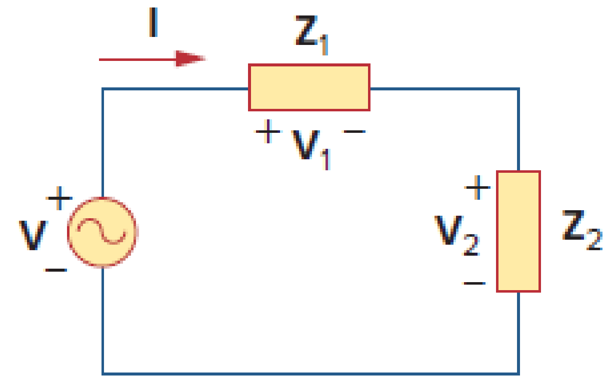
$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N)$$

$$\mathbf{Z}_{eq} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N \Rightarrow \boxed{\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N}$$

- The current through the impedances is

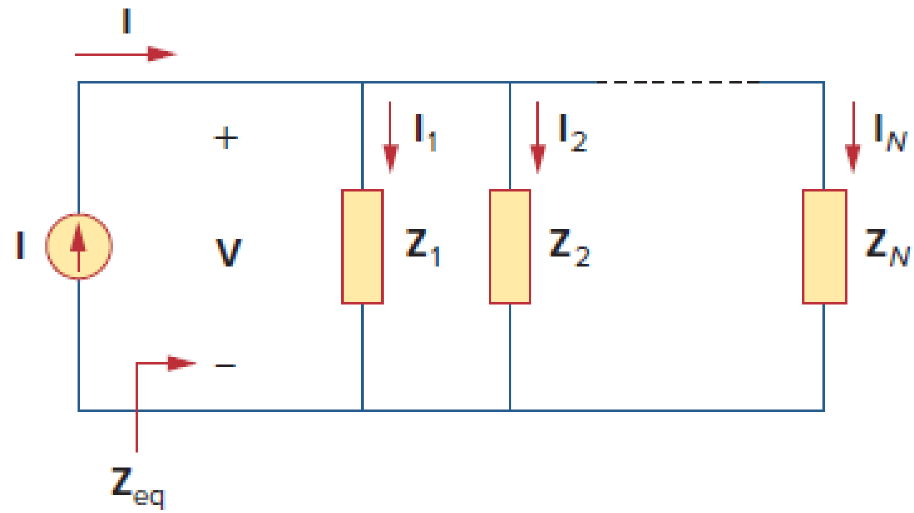
$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

- The Voltage drop across the impedances is(V.D.R)



$$\boxed{\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}}$$

II. Impedance in Parallel



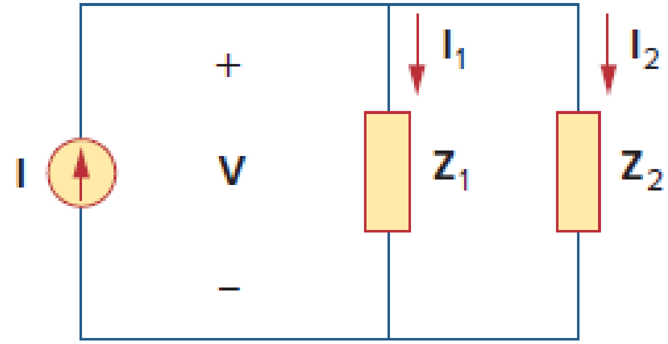
$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right)$$

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}$$

and the equivalent admittance is

$$\mathbf{Y}_{eq} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_N$$

Current division Rule

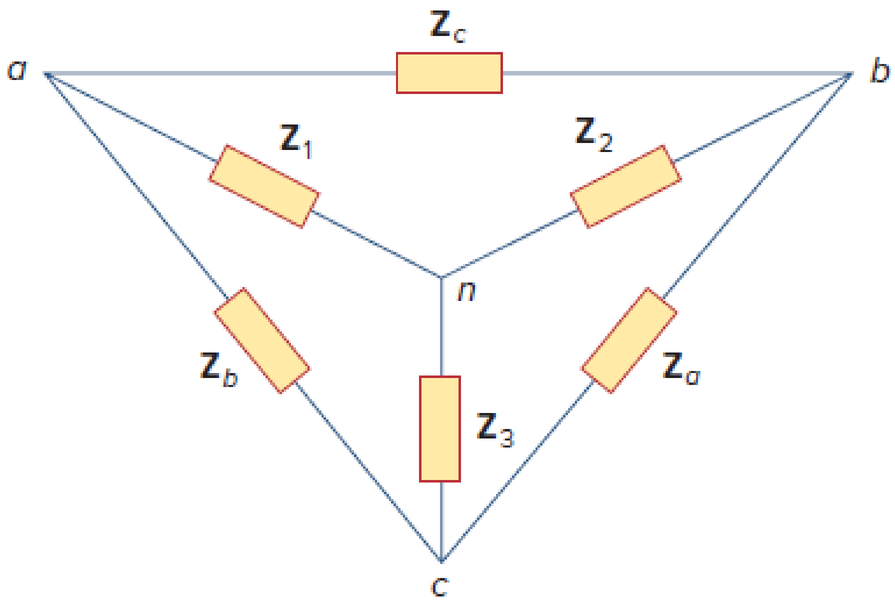


$$\mathbf{Z}_{eq} = \frac{1}{\mathbf{Y}_{eq}} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{1/\mathbf{Z}_1 + 1/\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = \mathbf{I}_1\mathbf{Z}_1 = \mathbf{I}_2\mathbf{Z}_2$$

$$\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}, \qquad \mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$$

The delta-to-wye and wye-to-delta transformations



Y-Δ Conversion:

$$Z_a = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1}$$

$$Z_b = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2}$$

$$Z_c = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3}$$

Δ-Y Conversion:

$$Z_1 = \frac{Z_bZ_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_cZ_a}{Z_a + Z_b + Z_c}$$

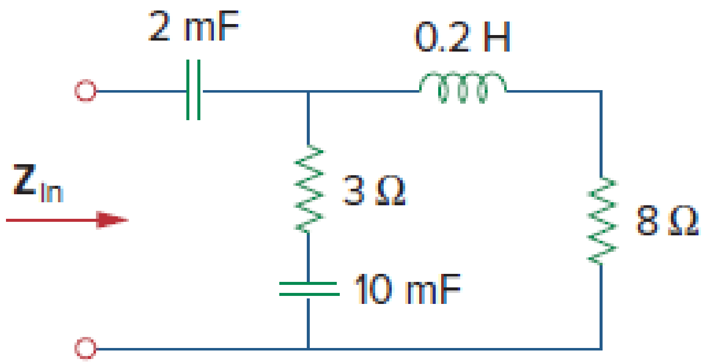
$$Z_3 = \frac{Z_aZ_b}{Z_a + Z_b + Z_c}$$

When a Δ-*Y* circuit is balanced, (equal Impedance)

$$Z_Y = Z_1 = Z_2 = Z_3 \text{ and } Z_\Delta = Z_a = Z_b = Z_c.$$

$$Z_\Delta = 3Z_Y \qquad \text{or} \qquad Z_Y = \frac{1}{3} Z_\Delta$$

Example: Find the input impedance of the circuit in the Figure below. Assume that the circuit operates at $\omega = 50$ rad/s.



Solution:

Let

$\mathbf{Z}_1$ = Impedance of the 2-mF capacitor

$\mathbf{Z}_2$ = Impedance of the 3-Ω resistor in series with the10-mF capacitor

$\mathbf{Z}_3$ = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

Then

$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

- **The input impedance is**

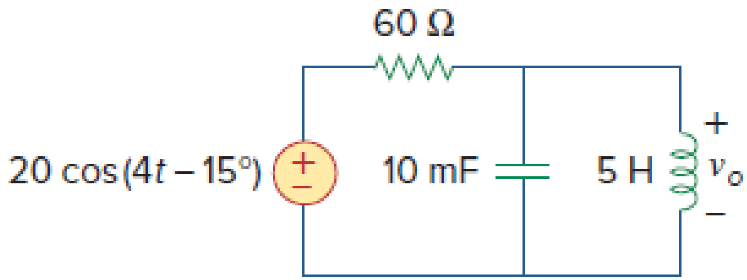
$$\mathbf{Z}_{in} = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3 = -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8}$$

$$= -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j10 + 3.22 - j1.07\ \Omega$$

Thus,

$$\mathbf{Z}_{in} = 3.22 - j11.07\ \Omega$$

Example: Determine $v_o(t)$ in the circuit of Figure below.



Solution:

The transformation (time domain circuit in to the phasor domain equivalent Circuit)

$$v_s = 20\cos(4t - 15^\circ) \quad \Rightarrow \quad \mathbf{V}_s = 20\underline{/-15^\circ}\text{ V}, \qquad \omega = 4$$

$$10\text{ mF} \quad \Rightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4 \times 10 \times 10^{-3}} = -j25\ \Omega$$

$$5\text{ H} \quad \Rightarrow \quad j\omega L = j4 \times 5 = j20\ \Omega$$

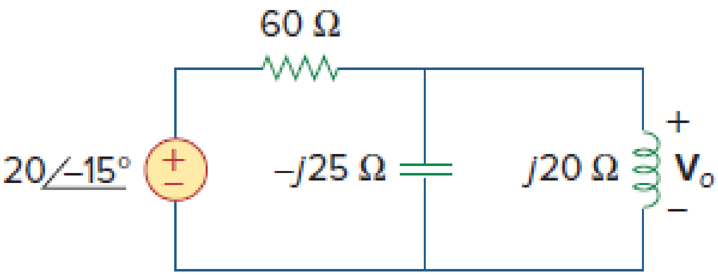


Figure: The frequency domain equivalent of the circuit

let

$\mathbf{Z}_1$ = Impedance of the 60-Ω resistor

$\mathbf{Z}_2$ = Impedance of the parallel combination of the 10-mF capacitor and the 5-H inductor

Then $\mathbf{Z}_1 = 60\ \Omega$ and

$$\mathbf{Z}_2 = -j25 \parallel j20 = \frac{-j25 \times j20}{-j25 + j20} = j100\ \Omega$$

By the voltage-division principle,

$$\mathbf{V}_o = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = \frac{j100}{60 + j100}(20\underline{/-15^\circ})$$

$$= (0.8575\underline{/30.96^\circ})(20\underline{/-15^\circ}) = 17.15\underline{/15.96^\circ}\ \text{V}$$

We convert this to the time domain and obtain

$$v_o(t) = 17.15\cos(4t + 15.96^\circ)\ \text{V}$$

AC Bridges

- An ac bridge circuit is used in measuring the inductance *L* of an inductor or the capacitance *C* of a capacitor.
- It is similar in form to the Wheatstone bridge for measuring an unknown resistance
- Galvanometer(ac ammeter or voltmeter)

❖ The bridge is *balanced*

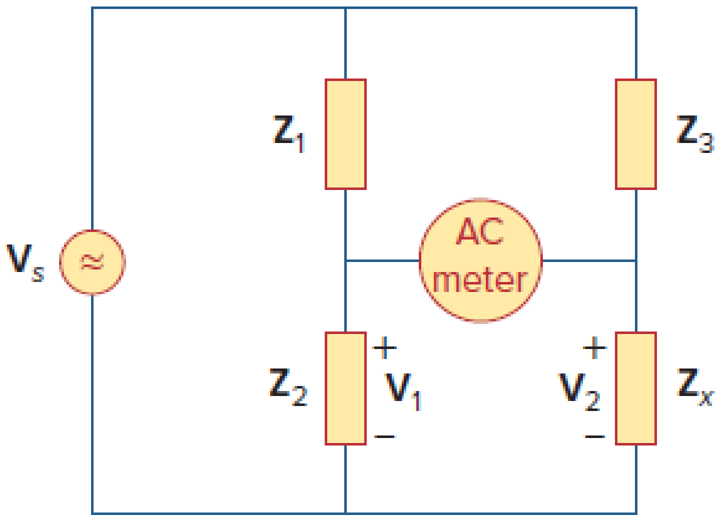
- no current flows through the meter

$$\mathbf{V}_1 = \mathbf{V}_2.$$

$$\mathbf{V}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = \mathbf{V}_2 = \frac{\mathbf{Z}_x}{\mathbf{Z}_3 + \mathbf{Z}_x}\mathbf{V}_s$$

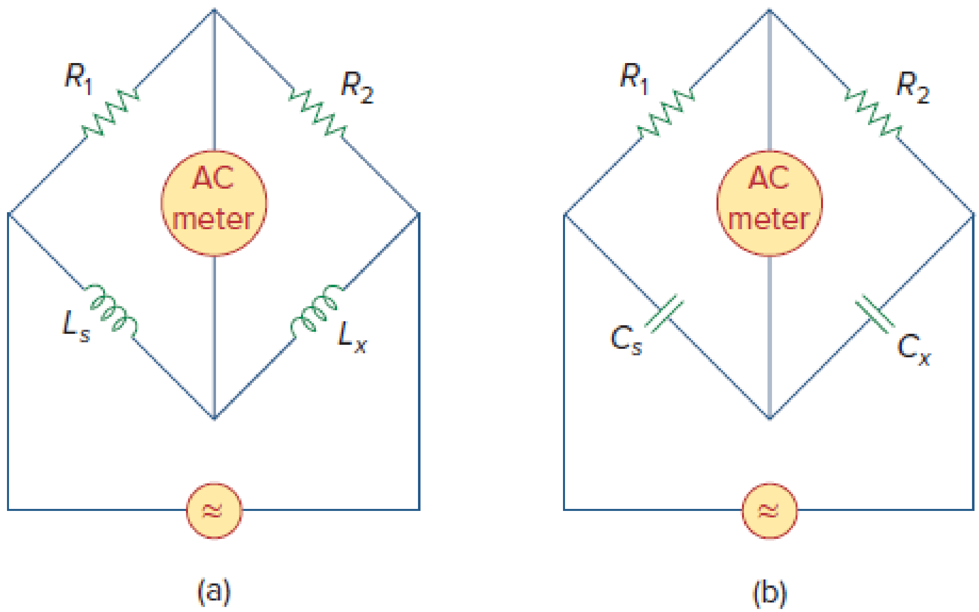
Thus,

$$\frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{\mathbf{Z}_x}{\mathbf{Z}_3 + \mathbf{Z}_x} \quad \Rightarrow \quad \mathbf{Z}_2\mathbf{Z}_3 = \mathbf{Z}_1\mathbf{Z}_x$$



$$\mathbf{Z}_x = \frac{\mathbf{Z}_3}{\mathbf{Z}_1}\mathbf{Z}_2$$

- Specific ac bridges for measuring *L* and *C*

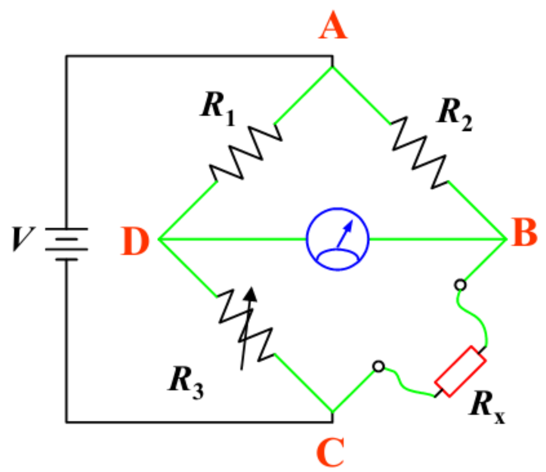


$$C_x = \frac{R_1}{R_2} C_s$$

$$L_x = \frac{R_2}{R_1} L_s$$

Figure: Specific ac bridges: (a) for measuring *L*, (b) for measuring *C*

Exercise: A Wheatstone bridge has a ratio arm of 1/100 (R_2/R_1).At first balance, R_3 is adjusted to 1000.3 Ω. The value of *Rx* is then changed by the temperature change, the new value of R_3 to achieve the balance condition again is 1002.1 Ω. Find the change of *Rx* due to the temperature change.

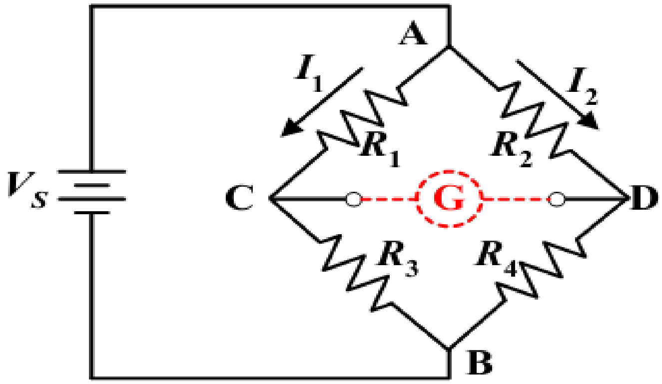


SOLUTION At first balance: $$R_x\text{old} = R_3 \frac{R_2}{R_1} = 1000.3 \times \frac{1}{100} = 10.003\ \Omega$$

After the temperature change: $$R_x\text{new} = R_3 \frac{R_2}{R_1} = 1002.1 \times \frac{1}{100} = 10.021\ \Omega$$

Therefore, the change of R_x due to the temperature change is 0.018 Ω

D.C bridge *(Off balance, $V_{CD} \neq 0$)*



Thévenin Voltage (V_{TH})

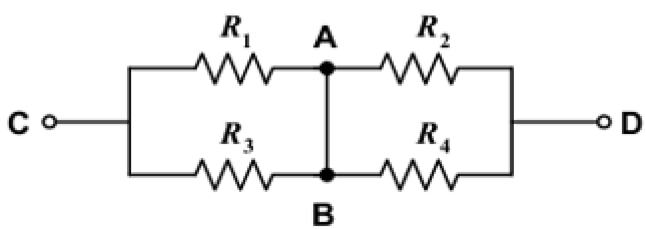
$$V_{CD} = V_{AC} - V_{AD} = I_1 R_1 - I_2 R_2$$

where $I_1 = \dfrac{V}{R_1 + R_3}$ and $I_2 = \dfrac{V}{R_2 + R_4}$

Therefore $$V_{TH} = V_{CD} = V\left(\frac{R_1}{R_1 + R_3} - \frac{R_2}{R_2 + R_4}\right)$$

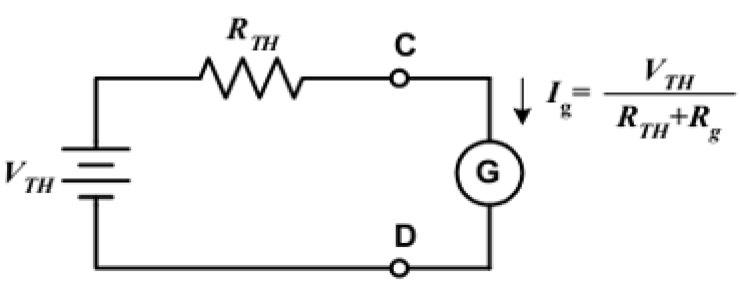
Using Thévenin circuit reducing method, *thevenin resistance, galvanometer current and galvanometer resistance* could be calculated as follows.

Thévenin Resistance (R_{TH})



$$R_{TH} = R_1 // R_3 + R_2 // R_4$$

Completed Circuit



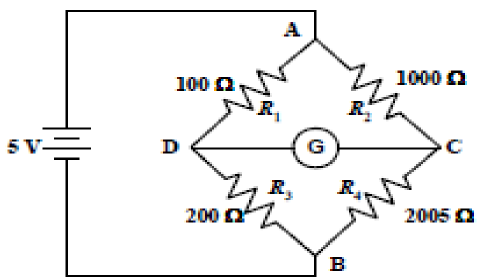
$$I_g = \frac{V_{TH}}{R_{TH} + R_g}$$

where I_g = the galvanometer current
R_g = the galvanometer resistance

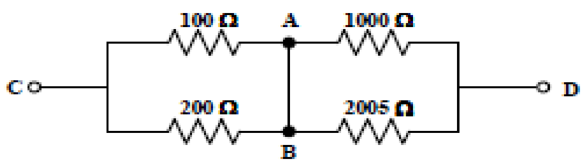
Exercise:

A Wheatstone bridge with values of the bridge elements R1 =100Ω, R2 = 1000Ω, R3=200Ω and R4 = 2005Ω. The battery voltage is 5V and its internal resistance negligible. The galvanometer has a current sensitivity of 10mm/μA and an internal resistance of 100Ω. Calculate the deflection of the galvanometer caused by the 5Ω unbalance in arm RX

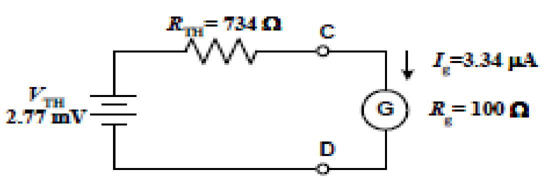
SOLUTION The bridge circuit is in the small unbalance condition since the value of resistance in arm BC is 2,005 Ω.



(a)



(b)



(c)

Thévenin Voltage (V_{TH})

$$V_{TH} = V_{AD} - V_{AC} = 5\text{ V} \times \left(\frac{100}{100+200} - \frac{1000}{1000+2005} \right)$$
$$\approx 2.77\text{ mV}$$

Thévenin Resistance (R_{TH})

$$R_{TH} = 100 \,//\, 200 + 1000 \,//\, 2005 = 734\ \Omega$$

The galvanometer current

$$I_g = \frac{V_{TH}}{R_{TH} + R_g} = \frac{2.77\text{ mV}}{734\ \Omega + 100\ \Omega} = 3.32\ \mu\text{A}$$

Galvanometer deflection

$$d = 3.32\ \mu\text{A} \times \frac{10\text{ mm}}{\mu\text{A}} = 33.2\text{ mm}$$

Thank You For Your Attention!

Questions?